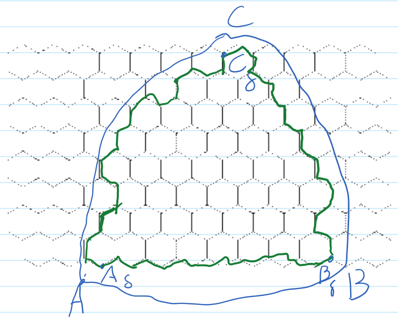
Another observable for percolation:
- specific for percolation

global (as opposite to parafermionic observable).
- only proven to converge for one specific model

site percolation on hexagonal lattice



Ω - simply connected domain,

$A, B, C \in \partial\Omega$

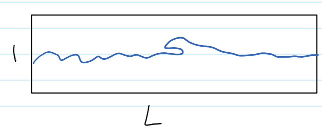
Ω_δ ~ δ-approximation by δ-Hexagonal Lattice.

$A_\delta, B_\delta, C_\delta$ – vertices closest to A, B, C correspondingly.

History:


John Cardy

Cardy's formula:



$1 \times L$

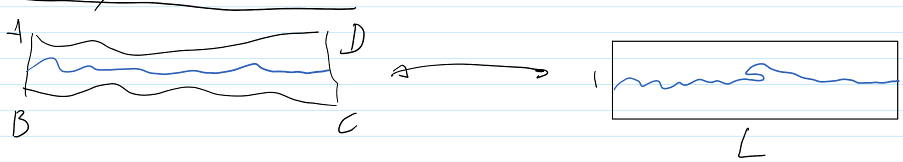
Probability of left-right crossing in a (discrete approximation to) $1 \times L$ rectangle is

$$\frac{3\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}\left(\frac{L^2}{1+L^2}\right)^{\frac{1}{3}} {}_1F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{L^2}{1+L^2}\right).$$

Here

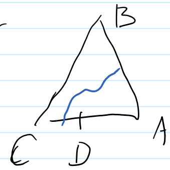
$${}_1F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, t\right) = \sum_{n=0}^{\infty} \frac{\frac{1}{3}\left(\frac{1}{3}+1\right)\cdots\left(\frac{1}{3}+n-1\right)\frac{2}{3}\left(\frac{2}{3}+1\right)\cdots\left(\frac{2}{3}+n-1\right)}{\frac{4}{3}\left(\frac{4}{3}+1\right)\cdots\left(\frac{4}{3}+n-1\right)} \frac{t^n}{n!}$$

Conformally invariant.



Lennart Carleson

Carleson form:

$P\Big($  $\Big) = \dfrac{|CD|}{|AC|}$

Cardy – Smirnov observable: discrete characterization of complexification:

$E_{A,\delta}$ $\qquad E_{B,\delta}$ $\qquad E_{C,\delta}$

$H_{A,\delta} := \mathbb{P}(E_{A,\delta})$ $\quad H_{B,\delta} := \mathbb{P}(E_{B,\delta})$ $\quad H_{C,\delta} := \mathbb{P}(E_{C,\delta})$

Initially defined on vertices, but can be extended continuously to the whole Ω_δ.
(affinely)

Lucio Russo

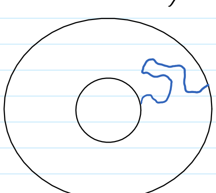
Paul Seymour

A priori uniform regularity of $H_{\cdot,\delta}$:

Lemma (Russo – Seymour – Welsh)

Let $A_z(r,2r)$ – annulus centered at z, inner radius r, outer radius $2r$, $r >> \delta$ ($r \geq 100\delta$). Then $\exists\, q > 0$.

$1-q > P(\exists$ blue crossing of $A_z(r,2r)) > q$

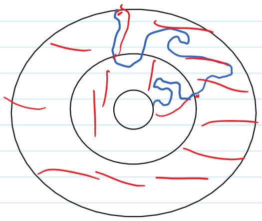


Proof. No proof here ☹

Corollary. $\exists \beta > 0$: $P(\exists$ blue crossing of $A_z(r,R)) \lesssim \left(\frac{r}{R}\right)^\beta$

provided $R \geq 2r$, $r \geq 100\delta$.

Proof



We need to cross $\log_2 \frac{R}{r}$ annuli of ratio 2, so

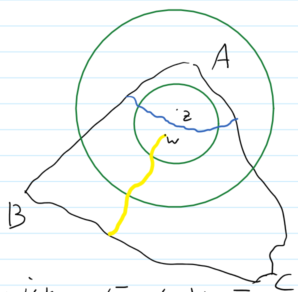
$$P(\text{crossing}) \lesssim q^{\log_2 \frac{R}{r}} = \left(\frac{r}{R}\right)^{\beta},$$

$$\beta = \log_2 \frac{1}{q}$$

Corollary. $\exists\, K_\Omega, \beta$: $|H_{\cdot,\delta}(z) - H_{\cdot,\delta}(w)| \le K_\Omega |z-w|^{\beta}$

Proof. Pick R, so that $\forall z$ $B(z,R)$ intersects at most 2 sides. Let $\sqrt{|z-w|} < \frac{R}{2}$.

Look at $E_{A,\delta}(z) \setminus E_{A,\delta}(w)$.



There is a yellow crossing from w to BC.

So there is always blue or yellow crossing of $A_z(\sqrt{|z-w|}, R)$.

same with $E_{A,\delta}(w) \setminus E_{A,\delta}(z)$.

So $\left|H_{A,\delta}(z) - H_{A,\delta}(w)\right| \lesssim \left(\frac{\sqrt{|z-w|}}{R}\right)^{\beta}$

By Arzella-Ascoli: pre-compactness. Just need to see that all subsequential limits are the same.

For an oriented edge $e = \vec{xy}$,

let $H_\delta(e) := \frac{H_\delta(x) + H_\delta(y)}{2}$ — average

$\partial_e H := H(y) - H(x)$ — "derivative."

$P_{A,\delta}(e) := P(E_{A,\delta}(y) \setminus E_{A,\delta}(x))$ — probability that y is separated from BC but not x. Same for B, C.

Observe: $\partial_e H_{\cdot,\delta} = P_{\cdot,\delta}(e) - P_{\cdot,\delta}(-e)$.

Define: $\tau := e^{2\pi i/3}$, for an edge $e = \vec{xy}$, denote two other edges from x as $\tau . e$, $\tau^2 . e$

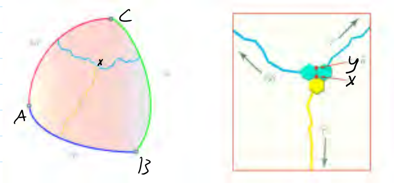
$\tau . e$ — x — e, $\tau^2 . e$

Theorem (Smirnov) $P_{C,\delta}(e) = P_{A,\delta}(\tau . e) = P_{B,\delta}(\tau^2 . e)$

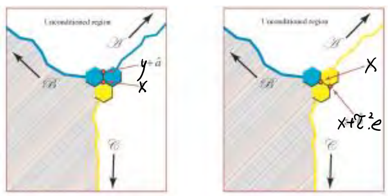
(Another version of discrete Cauchy-Riemann)

Proof

$E_{C,\delta}(y) \setminus E_{C,\delta}(x)$:



Take the highest blue crossing from e to AC.
Take the right most yellow crossing from $\tau^2 e$ to AB.



Recolor!

Get configuration of the same probability from $E_{B,\delta}(x+\tau^2 e)\setminus E_{B,\delta}(x)$.

$H_\delta(z) := H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z)$

$S_\delta(z) := H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z)$.

Let h and s be some subsequential limits.

Theorem. $h : (\Omega, A, B, C) \to (\Delta, 1, \tau, \tau^2)$ - conformal

(Δ is the triangle $(1, \tau, \tau^2)$)

$s \equiv 1$.

Corollary. $H_{A,\delta} + H_{B,\delta} + H_{C,\delta} \to 1$ (not obvious!)

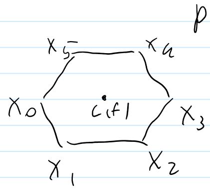
So

$H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z) \xrightarrow[\delta\to 0]{} h(z)$

$H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z) \xrightarrow[\delta\to 0]{} 1$

Observe $H_{A,\delta} \to \frac{2\operatorname{Re} h + 1}{3}$, so on AC - agrees with Cardy formula in Carleson form.

Proof. Let us start with integration by parts.



Let f be a face, $c(f)$ - its center, $x_0, \dots x_5$ - its vertices.

Then $\oint_{\partial f} H_\delta(x) = \sum_{e \in \partial f} e\, H_\delta(e) = \sum_{k=0}^{5} (x_{k+1} - x_k) \frac{H(x_{k+1}) + H(x_k)}{2} =$

$\oint$ of continuous integral of extended function.

$\sum_{k=0}^{5} (x_{k+1} - c(f)) \frac{H(x_{k+1}) + H(x_k)}{2} -$

$\sum_{k=0}^{5} (x_k - c(f)) \frac{H(x_{k+1}) + H(x_k)}{2} =$

$\sum_{k=0}^{5} (x_k - c(f)) \frac{-H(x_{k+1}) + H(x_k) + H(x_k) - H(x_{k-1})}{2} =$

$\sum_{k=0}^{5} \left(\frac{x_k + x_{k+1}}{2} - c(f)\right)\left(H(x_{k+1}) - H(x_k)\right) =$

$\sum_{k=0}^{5} \frac{e_k^*}{2} \partial_{e_k} H$ (e^* - dual edge to e) $=$

$\sum_{k=0}^{5} \frac{e_k^*}{[illegible]} \left(P_A(e) + \tau P_B(e) + \tau^2 P_C(e) - P_A(-e) - \tau P_B(-e) \dots \right)$

$$\sum_{k=0} \frac{e_k^*}{2} \partial_{e_k} H \quad (e^* - \text{dual edge to } e) =$$

$$\sum_{k=0}^{5} \frac{e_k^*}{2} \Big(P_A(e_k) + \tau P_B(e_k) + \tau^2 P_C(e_k) - P_A(-e_k) - \tau P_B(-e_k) - \tau^2 P_C(-e_k)\Big).$$

Let now γ be a smooth curve, γ_δ - its discretization,

Then

$$\oint_{\gamma_\delta} H_\delta = \sum_{e \text{ inside } \gamma_\delta} e^* \overbrace{\big(P_A(e) + \tau P_B(e) + \tau^2 P_C(e)\big)}^{=\, I} +$$

$$\underbrace{\sum_{e \subset \gamma_\delta} \partial_e H \cdot \frac{e^*}{2}}_{=\, II}. \quad \text{Know apriori: } \partial_e H \leq k\delta^\beta.$$

So $|II| \leq k\delta^\beta \cdot \text{length}(\gamma_\delta) \cdot \delta \lesssim \delta^\beta$ since $\text{length}|\gamma_\delta| \lesssim \frac{1}{\delta}$.

I: regroup edges around vertices inside γ_δ. So

$$I = \sum_{v \text{ inside } \gamma_\delta} e^* \big(P_A(e) + \tau P_B(\tau \cdot e) + \tau^2 P_C(\tau^2 \cdot e)\big)$$

$$\underset{\text{by Smirnov's Thm}}{=} \sum e^* \big(P_A(e) + \tau^2 P_A(e) + \tau^4 P_A(e)\big) = 0$$

$$\oint_{\gamma_\delta} H_\delta \lesssim \delta^\beta.$$

So if $h = \lim_{n\to\infty} H_{\delta_n}$, then

$$\oint_\gamma h = \lim_{n\to\infty} \oint_{\gamma_{\delta_n}} H_{\delta_n} = 0.$$ By Morera, h is analytic

Exactly the same computation for s.

So S is real, analytic $\Rightarrow S \equiv$ const.

$s(A) = 1 \Rightarrow S \equiv 1$.

Boundary conditions for h:

On $[BC]$: $h_A(z) = 0 \Rightarrow h_B(z) + h_C(z) = 1$.

Thus $h(z) = \tau h_B(z) + \tau^2 h_C(z) \in [\tau, \tau^2]$

Thus $h: [BC] \to [\tau, \tau^2]$, $h(B) = \tau$, $h(C) = \tau^2$.

$[A, B] \to [1, \tau]$

$[C, A] \to [\tau^2, 1]$.

h is also increasing on each side (since h_A, h_B, h_C monotone on each side).

<u>Lemma</u>. Let $h: \Omega_1 \to \Omega_2$ - analytic, γ - boundary of Ω_2, $h: \partial\Omega_1 \to \gamma$ preserves orientation. Then $h: \Omega_1 \to \Omega_2$ - conformal.

<u>Proof</u> (for Jordan $\partial\Omega_1$ for now)

Let $\partial\Omega_1$ - Jordan curve. Take $a \in \Omega_2$.
Then the number of solutions to $h(z) = a$ is equal to winding number of $h(\partial\Omega_1) = \gamma$ around a, i.e. 1 ∎